APPLICATION OF TVD FLUX LIMITING SCHEMES IN SIMULATION OF MULTIPHASE FLOW THROUGH POROUS MEDIA

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ABSTRACT
Frequently, the simulation of fluids flow through porous media uses single point upwind (upstream) weighting for approximating the transmissibility terms in mass-balance equations of each phase. This approximation gives a significant level of numerical dispersion that spreads the saturation front over many grid blocks. In order to correct this situation, especially when we have unstable flow processes or heterogeneous porous media, a better solution is using the high-order Godunov methods. Harten, point out two classes of Godunov schemes: TVD (Total Variation Diminishing) and ENO (Essentially Non-Oscillatory), both related to the total variation (TV) property. In this paper we use only the TVD schemes with different flux limiters which avoid the spurious oscillations associated with the classical second order schemes. The limiters used here are min-mod, Roe, van Leer, Chakravarty-Osher. If we have two phases and these are immiscible, the flow through porous media can be described by fractional flow. Also in this situation, if the capillary pressures and gravitational forces are neglected combining the two balance equations written for each phase, results a non-linear hyperbolic equation in saturation, well known as Buckley-Leverett equation. This equation can be used for model validation. The results provided by each numerical scheme applied for multi-phase fluid flows through porous media are compared with classical approach. Results show a significant improving of solution accuracy, but obtained with more computational effort.

KEYWORDS
Godunov Methods, TVD schemes, high order methods.

NOMENCLATURE

- $A$ [m$^3$/day] antidifusive flux
- $B_f$ [-] volume factor
- $f$ [m$^3$/day] flux
- $F$ [m$^3$/day] numerical flux
- $g$ [m/s$^2$] gravity
- $J$ Jacobian matrix
- $k$ [mD] absolute permeability
- $kr$ [-] relative permeability
- $p$ [bar] pressure
- $p_c$ [bar] capillary pressure
- $R$ residual vector
- $Rs$ gas-oil solution ratio
- $S$ [-] phase saturation
- $t$ [days] time
- $T$ geometrical transmissibility
- $u$ [-] nodal values
- $x$ horizontal direction
- $z$ [m] depth
- $\eta = \frac{\Delta t}{\Delta x}$ mesh ratio
- $\lambda = \frac{k \cdot \phi}{\mu} \cdot B$ fluid transmissibility
- $\mu$ [cP] viscosity
- $\phi$ [-] porosity
- $\psi(r)$ [-] flux limiter

Subscripts and Superscripts

- $f$ fluid phase
- $w$ water phase
- $o$ oil phase
- $g$ gas phase
- $n,n+1$ time level
- $k$ horizontal index of grid
- $ref$ reference values

ABBREVIATIONS

- TVD Total Variation Diminishing
- TV Total Variation
1. INTRODUCTION

Predictions of reservoir performances are important because they provide the data for planning, design and operation of the reservoir. A useful tool for reservoir behavior studies is numerical reservoir simulation which allows detailed predictions of reservoir performance studies is numerical reservoir simulation operation of the reservoir. A useful tool for reservoir because they provide the data for planning, design and

Predictions of reservoir performances are important

for each phase in the system. These equations include

the effects of viscous, capillary and gravitational

forces and allow the gas phase to be soluble in oil

phase. This kind of model is well known as black-oil

model. This is the most used model in reservoir simu-

lation. The black oil model assumes that the reservoir

fluids are composed by three components (water, oil and
gas) and three phases (water, oil and gas). The phases

flow occurs at constant temperature. The

water phase contains only the water component.

There are no phase transfer between oil phase and

water phase. The oil component can not exist in gas

phase, but the gas component can be dissolved in oil

phase. The mass transfer between the gas and oil is
described by the gas-oil solution ratio (Rs). The

reservoir phase proper-ties are determined by the

PVT analyses and are based on the properties in

stock-tank (surface) conditions.

Simulation of multiphase flow requires powerful

techniques because it deals with a system of coupled

nonlinear equations. A classical simplified description

of the simultaneous immiscible incompressible flow

of oil and water is based on the hyperbolic Buckley-

Leverett equation for the saturation.

Conventional numerical solvers for black-oil models

use finite differences, finite volumes or finite element

methods. Finite differences methods are most used

in numerical reservoir simulation because of their

simplicity and relative good accuracy even in complex

reservoir case.

A reliable numerical model must be able to take

into account the complexity of reservoir arisen from

the complex structure of fluids which flow through

reservoir, the complexity of reservoir geometry, and

reservoir heterogeneity.

An important feature in numerical simulation is

order of accuracy and also stability. The classical finite

difference method is one-point upstream weighting

which achieves first order of accuracy in space and
time and therefore suffers from large amount of nu-

merical diffusion. Therefore is more convenient to

use the second order methods. But the classical second

order methods (Lax-Wendroff[4], Warming and Beam

[11]) show spurious oscillations. Therefore, the new

schemes have appeared that reduce the numerical
diffusion and reduce the un-physical oscillations of

results. These schemes are based on Godunov high

order schemes [1] which use the solution of Riemann

problem at cell edges. Another kind of schemes are

those based on Total Variation Diminished (TVD)

property introduced by Harten [3] using flux limiters

(Sweby [9]). Many schemes which incorporate some

form of flux limiters was proposed for solving the

flux-conservative equation, Roe [6], van Leer [10],

Chakravarty and Osher [5].

An alternate approach to deriving a second order

accurate TVD Scheme is Monotonised Upstream

Centered Scheme for Conservation Laws known as

MUSCL originally due to Van Leer[10] described by

Goodman and LeVeque [2] which can be viewed as

second order generalisation Godunov’s scheme. MUSCL

scheme solves a linearised Riemann problem at each

cell interface.

In reservoir simulation area, Rubin and Blunt [7]
have used the TVD flux limiters scheme. They have
develop a TVD mid- point scheme which now is

most used in this area.

2. MATHEMATICAL MODEL

The fluid in the reservoir is considered to be com-

posed by three phases: gas, oil, water. In classical

black-oil model only the gas phase is miscible with

oil phase, the miscibility is characterised by gas-oil

solution ratio (Rs) which is dependent only on pressure.

The governing equations for the incompressible

and viscous flow are in one dimension:

\[
\frac{\partial}{\partial x} \left[ \lambda_w \left( \frac{\partial p_w}{\partial x} - \rho_w g \frac{\partial z}{\partial x} \right) \right] = \frac{\partial}{\partial t} \left( \phi \frac{S_w}{B_w} \right) + q_w
\]

\[
\frac{\partial}{\partial x} \left[ \lambda_o \left( \frac{\partial p_o}{\partial x} - \rho_o g \frac{\partial z}{\partial x} \right) \right] = \frac{\partial}{\partial t} \left( \phi \frac{S_o}{B_o} \right) + q_o
\]

\[
\frac{\partial}{\partial x} \left[ R_{sg} \left( \frac{\partial p_s}{\partial x} - \rho_s g \frac{\partial z}{\partial x} \right) \right] +
\frac{\partial}{\partial x} \left[ \rho_c \left( \frac{\partial q_w}{\partial x} + \rho_w g \frac{\partial z}{\partial x} \right) \right] =
\frac{\partial}{\partial t} \left( \phi R_c \frac{S_w}{B_w} + \phi \frac{S_o}{B_o} \right) + R_s q_o + q_g
\]

\[
p_{cow} = p_o - p_w; \quad p_{cog} = p_o - p_g
\]

\[
S_w + S_o + S_g = 1
\]
3. FLUX LIMITING TECHNIQUE

The explanations of main numerical methods use the one-dimensional scalar conservation law hyperbolic equation as in Sweby [9]. Given scalar conservation law:

\[
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0
\]

(2)

Given scalar conservation law:

\[
0 = \frac{\partial}{\partial x} (u f) + \frac{\partial}{\partial t} u
\]

(3)

A numerical scheme can be derived integrating the conservative law in a regular grid over the box \([x_{k-1/2}, x_{k+1/2}][t_n, t_{n+1}]\).

\[
\int_{t_n}^{t_{n+1}} \int_{x_{k-1/2}}^{x_{k+1/2}} (u f) \, dx \, dt = \Delta x \Delta t \sum_k (u_{k+}^n - u_{k-}^n)
\]

(4)

This integration gives:

\[
u_{k+1}^n = u_k^n - \eta \left( F_{k+1/2}^n - F_{k-1/2}^n \right)
\]

(5)

where \(\eta\) is the mesh ratio, and \(u_k^n\) are nodal values

\[
u_k^n = \frac{1}{\Delta x} \int_{x_{k-1/2}}^{x_{k+1/2}} u(k \cdot \Delta x,n \cdot \Delta t) \, dx
\]

(6)

And \(F\) is a consistent numerical flux function

\[
F_{k+1/2}^n = \frac{1}{\Delta t} \int_{u_k^n}^{u_{k+1}^n} f(u(x_{k+1/2}, t)) \cdot \sigma \, dt
\]

(7)

where: \(\sigma\) is unit outward normal.

The bounds of solution variations are the important estimations for convergence proofs. Therefore the Total Variation, \(TV(u_{k+1}^n)\) of the solution is defined [3,9] by:

\[
TV(u_{k+1}^n) = \sum_k |u_{k+1}^n - u_{k}^n|
\]

(8)

An important class of method based on Total Variation are called TVD (Total Variation Diminishing) [3] and the approximative solutions are characterised by:

\[
TV(u_{k+1}^n) \leq TV(u^n)
\]

(9)

A flux limiter scheme is a higher resolution scheme based on TVD where the application of a low order scheme is supplemented by addition of limited antidiffusive flux [9]. This flux is a difference between the flux of a high order scheme and that of the low order scheme which has been limited so that it ensures the TVD property of resulting scheme.

The limiter is a non-negative function \(\psi\), so it maintain the sign of the antidiffusive flux and the limited antidiffusive flux is maximized in amplitude subject to the TVD constraints (see Sweby[9]).

\[
F_{k+1/2}^n + \psi(r_{k+1/2})A_{k+1/2}
\]

(10)

We take the limiter to be a function of consecutive gradients

\[
r_{k+1/2} = \frac{A_{k+1/2}}{A_{k+1/2}}
\]

(11)


The minmod and superbee limiters are:

\[
\psi(r) = \max(\min\{r, 1\}, 0)
\]

(12)

Van Leer limiter is:

\[
\psi(r) = \frac{|r| + r}{1 + |r|}
\]

(13)

The Chakravarty-Osher limiter is:

\[
\psi(r) = \max(0, \min(2r, 1) \min(r, \xi))
\]

(14)

Where \(1 \leq \xi \leq 2\) which generate a family of limiters. If \(\xi = 1\) then the Chakravarty-Osher limiter become the Roe’s minmod limiter.

4. THE MULTIPHASE FLOW CASE

Complex second order schemes produce accurate results for scalar-hyperbolic conservation law. When these schemes are applied to multiphase flow through porous media the mixed parabolic-hyperbolic natures of flow equations lead to complicate usage [7]. The parabolic and hyperbolic parts of equations are usually split and solved separately. This approach requires a great deal of code development to introduce these schemes into IMPES or fully implicit formulations [7].

In order to develop a numerical model for multiphase flow we consider a mixed parabolic-hyperbolic equations (1) and we must apply the flux limiter method presented in previous section to these equations that describe the flow of three phase fluid (water, oil and gas) through porous media. The flux of every phase can be written as:

\[
f_f = -k_f(S) \frac{\partial}{\partial x}(\rho_f - \rho_g g z)
\]

(15)

The relative permeabilities of phases \(k_{rf}\) are functions only of phase saturations, \(S_w\) and \(S_g\). Typically for two phase flowing we use the Brooks-Corey correlations for relative permeabilities:

\[
645
\]
The volume factor $B_f$ for each phase is considered linear function of pressure:

$$B_f = B_{ref}^f (1 + c_{ref}^f (p - p_{ref}))$$

where $B_{ref}^f$, $c_{ref}^f$ are volume and compressibility factors at reference pressure $p_{ref}$. The phase densities are given by (16):

$$\rho_w = \frac{\rho_w^{ref}}{B_w}$$
$$\rho_o = \frac{\rho_o^{ref} + R_s \rho_g^{ref}}{B_o}$$
$$\rho_g = \frac{\rho_g^{ref}}{B_g}$$

We consider as primary unknowns the pressure of oil phase, saturation of water phase and saturation of gas phase and we express all other properties function of these unknowns. Therefore the pressure of phases, water and oil will be:

$$p_w = p_o - p_{cow}$$
$$p_g = p_o + p_{cog}$$

The numerical model uses the IMPES (IMplicit Pressure Explicit Saturation) formulation [8]. When we use the TVD flux limiter we must use different limiter for each flux of phase. The flux limiter technique use the TVD mid-point scheme used by Rubin and Blunt [7]. Therefore, for a phase $f$ and node $k$, we can develop the following numerical model:

$$\Delta \Phi^{n+1} = \left[ \rho_w^{n+1} - \rho_o^{n+1} g z + \rho_c^{n+1} \right]_{k+1/2} - \left[ \rho_o^{n+1} g z + \rho_c^{n+1} \right]_k$$

$$\left( r_f \right)_k^{n+1} = -(kT)_k \left( \frac{k_f (S_f^2)}{\mu_f B_f^2} \right) \Delta \Phi^{n+1}$$

$$A_{k+1/2} = \frac{\left( r_f \right)_k^{n+1} - \left( r_f \right)_k^{n+1}}{2}$$

$$r_k = \frac{A_{k-1/2}}{A_{k+1/2}}$$

$$F_{k+1/2}^{n+1/2} = \left( r_f \right)_k^{n+1} + \psi \left( r_k \right) A_{k+1/2}$$

$$R_{f_k}^{n-1} = \left( \frac{\phi S_k}{B_f} \right)^{n+1} - \left( \frac{\phi S_f}{B_f} \right)^n + \left( \frac{E_{k+1/2}^{n+1/2} - E_{k-1/2}^{n+1/2}}{2} \right) = 0$$

The residual $R_{f_k}^{n+1}$ depend nonlinearly by oil pressure, water and gas saturations. Consequently, we use the Newton method for solving the nonlinear equations, that in matrix form can be written as:

$$J \Delta X = -R$$

Where $J$ is the Jacobian matrix, $\Delta X = X^{n+1} - X^n$ is the unknowns vector and $R$ is the residual vector.

In order to simplify the calculation[7] in Jacobian matrix we put only the first order flux, the limiter and antidiffusive flux are considered only for residual calculation, even if this procedure increase the number of Newton iterations.

5. NUMERICAL RESULTS

In order to validate the numerical model we initial consider a simple Buckley- Leveret problem, consi-dering a linear petroleum reservoir having the length equal with 500 meters. Using different TVD limiters such as minmod, superbee, van Leer, Chakravarty-Osher and different discretization’s number of nodes (10, 20, 50, 100), we compare the saturation distribution provided by each method.

The injection rate is 40 m$^3$/day, and the porosity of reservoir is considered 0.1. Consequently, figure 2 presents the comparison for the TVD limiter methods using 10 nodes, the figure 3 compares the limiters using 20 nodes, figure 4 and 5 compares the results provides by TVD methods for 50 respectively 100 nodes. The figure 2 shows a significant order of numerical disper-sion for all the methods (TVD or classical). This is a consequence of the relative small number of discritezation cells.

![Figure 2. Water saturation distribution at 2000 days case with 10 nodes](image)

The largest level of numerical dispersion was obtained from classical approach where the water saturation front is spread over the four cells. The smallest level of numerical dispersion was shown by Roe’s superbee limiter. This tendency is maintained even if we increase the number of cells.
Figure 3. Saturation distribution at 2000 days, case with 20 nodes

Figure 4. Water saturation distribution at 2000 days, case with 50 nodes

Figure 5. Water saturation distribution at 2000 days, case with 100 nodes

Figure 6. Water saturation distribution in time for superbee flux limiter

Figure 7. Water saturation at 2000 days comparison for different number of nodes and for superbee flux limiter

The augmentation of the number of cells leads to a reduction of numerical dispersion. So, for 100 cells even the classical scheme, without TVD flux limiters has a acceptable level of numerical dispersion.

The water saturation distribution for superbee flux limiter using 50 nodes at different times is presented in figure 6.

We can see that the augmentation of cells number reduces the level of numerical dispersion. For instance, the water saturation front for 10 cells is extended over 200 meters (40 % from the reservoir length) while for 100 cells the water saturation front has only 10 metres (2 % from the reservoir length). But, after a certain number of cells, increasing the number of cells do not causes a significant reduction of numerical dispersion level. We can observe that saturation front for 50 nodes is almost similar with the water saturation distribution provided by 100 nodes.

6. CONCLUSIONS

The paper presents a methodology for using the higher order schemes especially TVD flux limiter, to numerical reservoir simulation. The TVD flux limiting schemes normally are used only for hyperbolic equations but they can be adapted for parabolic partial
derivative equations. The main goal of TVD flux limiter schemes is to reduce the level of numerical dispersion that occur when the classical methods are used for developing the discrete approximation of partial differential equations. The TVD flux limiters applied to the Buckley-Leverett equation shows that results provided by Chakravarty-Osher limiter are similar with the result provided by Roe’s minmod limiter. The superbee limiter provides the results that present a lower numerical dispersion than van Leer or minmod limiters, having the least numerical dispersion. The van Leer limiter is less diffusive than minmod limiter. The CPU time required for computation increases from classical method toward van Leer methods. The minmod, Chakravarty-Osher and superbee limiters have provided approximately same values.

REFERENCES