EFFICIENT METHODS FOR 2D AND 3D STEADY AND UNSTEADY CONFINED FLOWS BASED ON THE SOLUTION OF THE NAVIER-STOKES EQUATIONS

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ABSTRACT

This invited paper presents efficient solutions of the steady and unsteady Navier-Stokes equations based on a finite difference formulation using artificial compressibility. They were obtained with methods recently developed for the efficient analysis of 2D and 3D unsteady confined flows with oscillating boundaries, needed in the study of fluid-structure interaction problems of engineering interest. These 2D and 3D methods reduce the flow problems to efficient solutions of scalar tridiagonal systems of equations by using a special decoupling procedure, which eliminates the pressure from the momentum equations with the aid of the continuity equation augmented by artificial compressibility. The numerical method is first validated for steady incompressible flows with multiple separation regions past a downstream-facing step, by comparison with previous computational and experimental results. Then, the method is used to obtain efficient solutions for several 2D and 3D unsteady flow problems with oscillating walls.

KEYWORDS
Aerodynamics, steady and unsteady flows, viscous flows, computational fluid dynamics.

1. INTRODUCTION

The steady and unsteady fluid-structure interaction problems are present in numerous engineering fields such as in aeronautics, gas and hydraulic turbines, pumps, thermo-fluid systems, nuclear reactors, bridges and tall buildings subjected to strong winds and other areas. This explains why the study of fluid-structure interaction problems, aeroelasticity and flow-induced vibrations, which involve multi-disciplinary analyses, recently received a topical interest.

The accurate analysis of fluid-structure interaction problems requires the simultaneous solution of the equations of the unsteady (or steady) flows and those of the deformation motion (often oscillatory) of the structure. As a result, the numerical methods of solution for the unsteady flows have to be characterized by an excellent computational efficiency, in addition to a very good accuracy. This requirement is made even more difficult by the complexity of the fluid flow problems, involving usually oscillating boundaries and flow separation regions, which require solutions of the Navier-Stokes equations capable to accurately capture these separation regions.

This paper presents efficient methods based on a finite difference formulations recently developed by this author for steady and unsteady flows with fixed and oscillating boundaries [11, 12, 14, 19, 20-22, 25, 26, 28-30]. Other efficient methods for the solution of steady and unsteady Euler and Navier-Stokes equations using finite volume, Lagrangian and spectral formulations [17, 23, 24, 27] are presented in other papers [15, 16].

2. EFFICIENT FINITE DIFFERENCE METHOD USING ARTIFICIAL COMPRESSIBILITY FOR SOLVING THE NAVIER-STOKES EQUATIONS

An efficient finite difference method is presented for the analysis of steady and unsteady viscous flows with oscillating boundaries, which may display multiple separation regions generated by large velocity gradients. In this method, the numerical problem is reduced to the efficient solution of scalar tridiagonal systems of equations by a special decoupling procedure based on the elimination of the pressure from the momentum equations by using the continuity equation augmented by artificial compressibility.
2.1. Problem formulation for 2D unsteady viscous flows with oscillating boundaries

The Navier-Stokes equations for 2D incompressible unsteady flows can be expressed in nondimensional form as

\[
\frac{\partial \mathbf{V}}{\partial t} + \mathbf{Q}(\mathbf{V}, p) = 0, \quad (1)
\]

\[
\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2)
\]

where \( x = x^*/H \) and \( y = y^*/H \) are dimensionless Cartesian coordinates (nondimensionalized with respect to a reference length \( H \)), \( \mathbf{V} = \{u,v\}^T \) is the dimensionless fluid velocity, (nondimensionalized with respect to a reference velocity \( U_0 \)), \( \mathbf{V} = \mathbf{V}^*/U_0 \), \( t = t^*/U_0/\nu \) is the nondimensional time, and \( \mathbf{Q}(\mathbf{V}, p) \), which includes the convective derivative, pressure and viscous terms, can be expressed in 2D Cartesian coordinates in the form

\[
\mathbf{Q}(\mathbf{V}, p) = (Q_u, Q_v, Q_p)^T, \quad (3)
\]

\[
Q_u = \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial p}{\partial x} - \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),
\]

\[
Q_v = \frac{\partial (vu)}{\partial x} + \frac{\partial (vv)}{\partial y} + \frac{\partial p}{\partial y} - \frac{1}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\]

in which \( p \) is the dimensionless pressure, (nondimensionalized with respect to \( \rho U_0^2 \), and \( Re = HU_0/\nu \) is the Reynolds number (\( \rho \) and \( \nu \) are the fluid density and kinematic viscosity).

On a boundary moving with the nondimensional velocity \( \mathbf{V}_b \), the no-slip condition can be expressed as \( \mathbf{V} = \mathbf{V}_b(x,y,t) \).

2.2. Method of solution

The main features of this method of solution are briefly presented in the following.

**Time dependent coordinate transformation.** For a rigorous implementation of the moving boundary conditions, the unsteady flow problem is solved in a fixed computational domain obtained from the physical fluid domain by a time-dependent coordinate transformation

\[
X = X(x,y,t), \quad Y = Y(x,y,t), \quad t = t. \quad (4)
\]

The specific form of this transformation depends on the type of moving boundaries, as shown further.

In the fixed computational domain, the Navier-Stokes and continuity equations can be expressed as

\[
\frac{\partial \mathbf{V}}{\partial t} + \mathbf{G}(\mathbf{V}, p) = 0, \quad (5)
\]

where

\[
\mathbf{D} \mathbf{V} = 0, \quad (6)
\]

\[
\mathbf{D} \mathbf{V} = C_1 \frac{\partial u}{\partial X} + C_2 \frac{\partial u}{\partial Y} + C_3 \frac{\partial v}{\partial X} + C_4 \frac{\partial v}{\partial Y}, \quad (7)
\]

\[
\mathbf{G}(\mathbf{V}, p) = \{G_u(u,v,p), G_v(u,v,p)\}^T \quad (8)
\]

\[
G_u(u,v,p) = C_1 \frac{\partial (uu)}{\partial X} + C_2 \frac{\partial (uv)}{\partial Y} + C_3 \frac{\partial (vu)}{\partial X} + C_4 \frac{\partial (vv)}{\partial Y} + C_5 \frac{\partial^2 u}{\partial X^2} + C_6 \frac{\partial^2 u}{\partial X \partial Y} + C_7 \frac{\partial^2 v}{\partial X^2} + C_8 \frac{\partial^2 v}{\partial X \partial Y} + C_9 \frac{\partial v}{\partial Y} \cdot (8a)
\]

\[
G_v(u,v,p) = C_3 \frac{\partial (vu)}{\partial X} + C_4 \frac{\partial (vv)}{\partial Y} + C_5 \frac{\partial^2 v}{\partial X^2} + C_6 \frac{\partial^2 v}{\partial X \partial Y} + C_7 \frac{\partial^2 u}{\partial X^2} + C_8 \frac{\partial^2 u}{\partial X \partial Y} + C_9 \frac{\partial u}{\partial Y}. \quad (8b)
\]

In these expressions, the coefficients \( C_i \) are defined corresponding to the variable velocity and pressure.

**Real time discretization.** First, a second-order three-point backward scheme in the form

\[
\frac{\partial \mathbf{V}}{\partial t} - \frac{3}{2} \mathbf{V}^{n+1} - 4 \mathbf{V}^n + \frac{1}{2} \mathbf{V}^{n-1} \big| (2\Delta t) \quad (7)
\]

is used to discretize the momentum equation (4a) in real time, where \( t^{n+1} = t^n - \Delta t \), \( t^n \) and \( t^{n+1} = t^n + \Delta t \), in which \( \Delta t \) represents the real time step. Thus, the Navier-Stokes and continuity equations (4) become

\[
\mathbf{V}^{n+1} + \alpha \mathbf{G}(\mathbf{V}^{n+1}, p^{n+1}) = \mathbf{F}^n, \quad (10)
\]

\[
\mathbf{D} \mathbf{V}^{n+1} = \mathbf{0}, \quad (11)
\]

where

\[
\alpha = 2\Delta t/3, \quad \mathbf{F}^n = (4\mathbf{V}^n - \mathbf{V}^{n-1})/3. \quad (12)
\]

**Pseudo-time relaxation procedure.** Equations (8) are then augmented by pseudo-time derivatives using the artificial compressibility concept introduced by Chorin [5], and expressed in the form

\[
\frac{\partial \mathbf{V}}{\partial \tau} + \mathbf{V} + \alpha \mathbf{G}(\mathbf{V}, \mathbf{p}) = \mathbf{F}^n, \quad (13)
\]

\[
\frac{\partial \mathbf{p}}{\partial \tau} + \mathbf{D} \mathbf{V} = \mathbf{0}, \quad (14)
\]

where \( \mathbf{V}(\tau) \) and \( \mathbf{p}(\tau) \) denote pseudo-functions corresponding to the variable velocity and pressure.
at pseudo-time $\tau$ between the real-time levels $t^n$ and $t^{n+1}$, where $\delta$ represents the artificially added compressibility; an optimum value for $\delta$ can be determined using the theory of characteristics as shown by Mateescu et al. [19, 25, 26].

An implicit Euler scheme is then used to discretize equations (10) between two consecutive pseudo-time levels $\tau^n$ and $\tau^{n+1}$ in the form

$$
(1 + \alpha \Delta \tau) \dot{V}^{n+1} + \alpha \Delta \tau G(V^{\ast n+1}, \dot{p}^{\ast n+1}) = F^n + \dot{V}^n, \quad (15)
$$

$$
\rho^{n+1} + (\Delta \tau/\delta) D \dot{V} = \dot{p}^n. \quad (16)
$$

These equations can further be recast in function of the pseudo-time variations $\Delta u = \dot{u}^{n+1} - \dot{u}^n$, $\Delta v = \dot{v}^{n+1} - \dot{v}^n$ and $\Delta p = \dot{p}^{n+1} - \dot{p}^n$ in the matrix form

$$
[ I + (\alpha \Delta \tau) (D_x + D_y)] \Delta f = S \Delta \tau, \quad (17)
$$

where $\Delta f = [\Delta u, \Delta v, \Delta p]^T$ is the unknown vector of the pseudo-time variations, $I$ is the identity matrix, and where the specific forms of the vector $S$ and of the $X$- and $Y$-differential operators $D_x$ and $D_y$ are defined by the geometric transformation (4).

A factored alternate direction implicit scheme (ADI) is used further to separate the matrix equation (17) in two successive sweeps in $X$ and $Y$ defined by the equations

$$
[I + (\alpha \Delta \tau) D_y] \Delta f^* = S \Delta \tau, \quad (18)
$$

$$
[I + (\alpha \Delta \tau) D_x] \Delta f = \Delta f^*, \quad (19)
$$

where $\Delta f^* = [\Delta u^*, \Delta v^*, \Delta p^*]^T$ is a convenient intermediate variable vector.

Spatial discretization. These equations are further spatially discretized by central differences on a stretched staggered grid, using hyperbolic sine and hyperbolic tangent stretching functions in the $X$- and $Y$-directions, respectively.

Decoupling procedure. A special decoupling procedure is used to eliminate the pressure from the momentum equations with the aid of the continuity equation, and to reduce the problem to the efficient solution of decoupled scalar tridiagonal system of algebraic equations. Details on the spatial discretization and the decoupling procedure can be found in our previous papers [19, 25, 26].

As a result of reducing the flow problem to the efficient solution of scalar tridiagonal systems, this method is characterized by excellent computational efficiency and accuracy, displayed in all steady and unsteady flow problems studied.

2.3. Method validation for steady flows with multiple separation regions

This method has been validated by comparison with previous computational and experimental results [2, 6, 9, 11, 37] obtained for the steady confined viscous flow with multiple separation regions past a downstream-facing step. The geometry of this confined flow is illustrated in Figure 1, with the value of the channel expansion ratio $H/h = 2$ taken in the computations for a meaningful comparison with previous results. A fully developed laminar flow with the mean velocity $U_0$ is considered at the inlet.

Figure 1. Geometry of the flow past a downstream-facing step.

In this case there is no need for the time-dependent coordinate transformation (3). The flow past the downstream-facing step is characterized by two separation regions developed on the lower wall (just behind the step) and on the upper wall due to large velocity gradients in these regions. This is shown in Figure 2 which presents a graphical illustration of the computed cross-channel axial velocity profiles at various axial locations along the computational domain ($-2 < x < 30$) for $Re=800$. The pattern of the streamline contours illustrating the recirculation regions near the upper and lower walls are also shown in the background of this figure.

The locations of the separation and reattachment points are commonly used as validation criteria for the computational and experimental results.

The main results obtained for the separation regions developed in this flow are shown in Table 1 (where $l_0$ denotes the upstream channel length before the step) and Figure 3. The results obtained with the present method [19, 20] were found in excellent agreement with Garling’s benchmark results [6] and with previous computational results obtained by other authors [9, 37] for various low Reynolds numbers between $Re\approx 400$ and 1200.
The comparison with experimental results obtained by Armaly et al. [2] and by Lee & Mateescu [11] (using a non-intrusive measurement technique based on multi-element hot-film sensors glued on the wall surface) indicated a good agreement for Reynolds numbers between 400 and 700, with a deteriorating agreement between Re=800 and 1200; this agreement deterioration was attributed by Gartling [6], Kim & Moin [9], Mateescu & Venditti [19] and Armaly et al. [2] to the three-dimensional effects occurring in the experimental flows due to the side walls, as opposed to the rigorous two-dimensional numerical analysis.

![Figure 2. Steady flow over a downstream-facing step (H/h = 2 and Re=800). Typical cross-channel velocity profiles at various axial locations and the streamline contours illustrating the recirculation regions.](image)

Table 1. Computed dimensionless length of separation on the lower wall, \( l_* \), compared with previous numerical and experimental results for various Reynolds numbers

<table>
<thead>
<tr>
<th></th>
<th>Re = 400</th>
<th>Re = 600</th>
<th>Re = 800</th>
<th>Re = 1000</th>
<th>Re = 1200</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Computational solutions for ( H/h = 2 )</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kim and Moin [9] ((l_0 = 0))</td>
<td>4.3</td>
<td>5.3</td>
<td>6.0</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Sohn [37] ((l_0 = 0))</td>
<td>4.1</td>
<td>5.2</td>
<td>5.8</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Gartling [6] ((l_0 = 0))</td>
<td>–</td>
<td>–</td>
<td>6.10</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Present solution ( l_0 = 0 ) (without upstream channel)</td>
<td>4.32</td>
<td>5.37</td>
<td>6.09</td>
<td>6.71</td>
<td>7.29</td>
</tr>
<tr>
<td>Present solution ( l_0 = 2 ) (with upstream channel)</td>
<td>4.12</td>
<td>5.17</td>
<td>5.90</td>
<td>6.53</td>
<td>7.11</td>
</tr>
<tr>
<td><strong>Experimental results for:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lee and Mateescu [11] (*)</td>
<td>4.1</td>
<td>5.21</td>
<td>6.45</td>
<td>7.4</td>
<td>8.4</td>
</tr>
<tr>
<td>Armaly et al. [2] (**)</td>
<td>4.3</td>
<td>5.8</td>
<td>7.1</td>
<td>8.1</td>
<td>8.9</td>
</tr>
</tbody>
</table>

\(*, **The experimental Reynolds numbers were slightly different in some cases (see also Figure 3).\)
3. SOLUTIONS OF 2D UNSTEADY FLOWS WITH OSCILLATING BOUNDARIES

This method has been used, after its validation for steady flows, to obtain efficient solutions for unsteady confined flows with oscillating boundaries and multiple separation regions. Thus, the unsteady viscous flows past a downstream-facing step have been thoroughly analyzed when a portion of length $Hl$ of the lower wall after the step executes harmonic oscillations defined as

$$g(x,t)=e(t)\sin(\pi x/l), \quad e(t)=A\cos(\omega t), \quad (20)$$

in which $A$ and $\omega=\omega^* H/U_0$ represent the dimensionless amplitude and the reduced frequency of oscillations ($\omega^*$ being the radian oscillation frequency).

In this case, the time-dependent coordinate transformation (3) can be expressed in the form

$$X = X', \quad Y = f(x,y,t), \quad t=t.' \quad (21)$$

where

$$f(x,t)=\begin{cases} \frac{h}{H} - \frac{h/H - y}{1-g(x,t)} & \text{for } x \in [0,l] \\ y & \text{for } x > l \end{cases} \quad (22)$$
The lengths of the separation regions on the lower and upper walls are obviously varying in time in this case due to the oscillatory motion of the lower wall.

The variations with time of the locations of the separation and reattachment points are shown in Figures 4 and 5 for various Reynolds numbers and amplitudes of oscillations.

Figure 4. Unsteady flow over a downstream-facing step with an oscillating floor ($H/h = 2$). Typical influence of the Reynolds number, Re, on the variation in time of the location of the separation and reattachment points on the upper and lower walls (for $\omega=0.05$ and $A=0.05$).
Figure 5. Unsteady flow over a downstream-facing step with an oscillating floor ($H/h = 2$). Typical influence of the amplitude of oscillations, $A$, on the variation in time of the location of the separation and reattachment points on the upper and lower walls (for $Re=400$ and $\omega=0.05$).

The influence of the Reynolds number on the locations of the separation and reattachment points is shown in Figure 4 for $\omega=0.05$ and $A=0.05$. As expected, the locations of the separations and reattachment points computed for $Re=600$, $700$ and $800$ display oscillations in time, and they are present on both the upper and lower walls during the entire oscillation cycle.
However, it was found that for Re=400 and 500 the upper recirculation region is present only during a portion of the oscillatory cycle and disappears for the rest of the cycle.

The influence of the oscillation amplitude is illustrated in Figure 5 for Re=400 and $\omega=0.05$. One can notice that the upper separation region is continuously present for smaller oscillation amplitudes, but disappears during a portion of the oscillation cycle for larger amplitudes, such as $A=0.05$ and 0.10.

A graphical illustration of the computed cross-channel profiles of the axial velocity at various locations ($-2 < x < 30$) along the computational domain is shown in Figure 6 at several moments during the oscillatory cycle. The streamline contours illustrating the upper and lower separation regions are also shown in Figure 6.

Figure 6. Unsteady flow over a downstream-facing step with an oscillating floor ($H/h=2$, Re=400, $\omega=0.05$ and $A=0.1$). Typical cross-channel velocity profiles at various axial locations along the computational domain and streamline contours illustrating the separation regions near the upper and lower walls at four different moments during the oscillatory cycle: (a) after 4 oscillation cycles (and after 5 oscillation cycles); (b) after 4.25 cycles; (c) after 4.50 cycles; (d) after 4.75 cycles.

4. SOLUTIONS OF 3D UNSTEADY FLOWS WITH OSCILLATING BOUNDARIES

The method presented in Section 2.2 has been extended for the solution of 3D unsteady viscous flows with oscillating boundaries.

Thus, unsteady flow solutions have been obtained for the first time for the 3D annular configurations with oscillating boundaries, such as that shown in Figure 7, in which a portion of the outer cylinder executes transverse harmonic oscillations, and the inner cylinder is fixed. The inner cylinder can have an annular downstream-facing step as shown in Figure 7. A fully developed laminar flow of mean velocity $U_0$ is considered at the inlet.

In this case, the 3D unsteady flow is referred to a system of dimensionless cylindrical coordinates, $x$, $r$ and $\theta$, which are nondimensionalized by the inlet annular clearance $h$ (shown in Figure 7). The transverse oscillatory motion of a portion of the outer cylinder is defined by the equation

$$he(t) = -he_0 \cos(\omega t),$$

where $e_0$ and $\omega = \omega^* H/ U_0$ represents the dimensionless amplitude and the reduced frequency of oscillations ($\omega^*$ being the radian oscillation frequency).
The unsteady flow problem is solved in a fixed computational domain defined by the time-dependent coordinate transformation

\[ x = x, \quad Z = (r - r_i)/H(0,t), \quad \theta = \theta, \quad t = t, \quad (24) \]

where

\[ H(0,t) = \sqrt{(r_i + 1)^2 - e^2(t)^2 - e(t)\cos \theta - r_i}, \quad (25) \]

for \( x \in (0, l) \) and \( H(0,t) = h \) for \( x \not\in (0, l) \). In the fixed computational domain, the moving and fixed portions of the outer cylinder is defined by \( Z = 1 \), and the fixed inner cylinder is defined by \( Z = 0 \) before the annular step and by \( Z = -(r_i - r_o) \) after the annular step.

In this computational domain, the solution of the flow problem is obtained following the same numerical procedures presented in Section 2.2. The details of the mathematical derivations (which are not presented here due to space limitation) can be found in our previous paper [22].

The unsteady flow solutions obtained for this flow configuration with oscillating outer wall are illustrated in Figure 8, which shows the axial variation of the amplitude of the reduced unsteady pressure on the outer cylinder at an azimuthal angle \( \theta = 7.5^\circ \) with respect to the plane of the oscillatory translation. Solutions are presented for the inner cylinder with or without an annular downstream-facing step, and for both cases with or without axial flow. The present solutions are compared in Figure 8 with the approximate solutions based on the mean-position analysis.

In Figure 9, the present 3D unsteady flow solutions obtained for the case of uniform inner cylinder are compared with the experimental results obtained by Mateescu et al. [22, 30].

![Figure 7. Geometry of the unsteady annular flow past a cylindrical inner body with an annular downstream-facing step and with the outer cylinder oscillating over the central portion of length \( l^* = hl \).](image)

![Figure 8. 3D unsteady annular flow between an oscillating outer cylinder and a fixed inner cylinder with or without a downstream-facing step. Typical axial variation of the amplitude of the reduced unsteady pressure on the outer cylinder at \( \theta = 7.5^\circ \): (a) with axial flow; (b) without axial flow. Comparison between: —— Present 3D unsteady solution; - - - Solution based on mean-position analysis.](image)
5. CONCLUSIONS

The paper presents 2D and 3D methods of solutions for unsteady incompressible flows based on the time-accurate solution of the Navier-Stokes equations. This method uses a three-point backward scheme for the real-time discretization, and a pseudo-time relaxation procedure, in which the continuity equation is augmented by an artificial compressibility, in order to advance the solution to a new real time level. A special procedure is used to decouple the momentum equations with the aid of the continuity equation augmented by artificial compressibility. Thus, the flow problem is reduced in this method to the efficient solution of scalar tridiagonal systems of equations, which contribute essentially to an excellent computational efficiency of the method. The spatial discretization is based on a central difference formulation on stretched staggered grids.

The method is successfully validated for steady flows with multiple separation regions past a downstream-facing step by comparison with previous experimental and computational results. After validation, the method has been used to obtain time-accurate solutions for 2D and 3D unsteady viscous flows with oscillating boundaries.

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flows. Aerospace Science and Technology Journal (accepted for publication)


